1 Introduction

The theory of NP-completeness was one of the first successful programs of *complexity theory*, a central field of theoretical computer science. In the domain of quantum computation, there is a direct analogue of NP called QMA. The theory of QMA-completeness has advanced much more slowly than the theory of NP-completeness. To make this a bit more precise, here's a fun set of data.

Cook published the first NP complete problem in 1971 [3]. Just one year later, Karp published a list of 21 NP-complete problems, [7] many of them coming from combinatorial properties of graphs. By contrast, the class QMA was introduced in 2001 by Kitaev [8]. Adam Bookatz published a survey in 2013 attempting to list all known QMA-complete problems. [2] Just like Karp, he finds exactly 21!

In 2007, Beigi and Shor had the following idea for generating QMA-complete problems: take a graph problem, interpret it as a problem about zero-error information theory, and then lift it to the quantum information setting. The had some success, defining their QMA-complete QUANTUM-2-CLIQUE problem. [1] The recent introduction of quantum graphs by Duan, Severini, and Winter [5] gives us machinery to make Beigi and Shor's program systematic. In particular, this viewpoint will lead us to the following computational problem.

Definition 1.1 (QUANTUM-k-COLORING)**.**

- Input: States $\rho_1, \rho_2 \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$.
- Promise: Either there exists some cptp map $T : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(C^k)$ such that
$$\mathrm{Tr}[(T \otimes I_B)\rho_1][(T \otimes I_B)\rho_2] \leq a,$$
or else for all such T,
$$\mathrm{Tr}[(T \otimes I_B)\rho_1][(T \otimes I_B)\rho_2] \geq b.$$
- Problem: Decide which is the case.

Conjecture 1.2. QUANTUM-k-COLORING *is* QMA-*complete*

The main goal of this article is to argue that this problem is a natural quantum generalization of k-coloring and that it feels like it should be QMA-complete.

1.1 Organization of the article.

The first section discusses the source-channel coding problem and is meant for anyone who knows what a graph is—no quantum necessary. The second section is a brief introduction to quantum graphs and should be readable by anyone with basic familiarity with quantum information. (I'll write more on quantum graphs later, hopefully including some things for people new to quantum.) The last section can be profitably read without the second section. It examines the computational complexity of the problems defined in the first section, reproduces the proof of Beigi and Shor's result from 2007, and motivates the conjecture mentioned in the introduction.

2 On Quantum Source-Channel Coding

The exposition in this section is very similar to exposition in Dan Stahlke's paper on the subject. [11] I hope to present it somewhat differently in order to motivate the computational complexity problem which is the goal of this article. However, if you want to see a much more polished exposition, go read Dr. Stahlke's paper! (As an additional plug, his paper proves some of its lemmas with the tensor network calculus, which is one of my most favorite physics things.)

2.1 Classical Channel Coding

Alice wants to send a message to Bob, but it is corrupted along the way. For example, suppose Alice's message is an integer between 0 and 4. Nature flips a fair coin and adds 1 (mod 5) to Alice's integer if heads and leaves it alone if tails. How much communication can Alice and Bob achieve in this situation?

Definition 2.1 (Classical channel)**.** A *classical noisy channel* is a map $T : A \times B \to [0,1]$ such that for all a, $T(a) = T(a, \cdot)$ is a probability distribution over B. That is, for fixed a, $\sum_b T(a,b) = 1$.

In the example we defined, we have $T(a,b) = \frac{1}{2}$ whenever $b - a \equiv 0, 1 \pmod 5$. In general, we can also think of T as a stochastic transition matrix from the state space over A to the state space over B.

Now we ask a question that motivates the rest of the article: given a fixed channel T, how can we test the capacity of T to perform communication tasks? To do this test, we'll introduce a testing party Charlie who referees a challenge for Alice and Bob. They'll be able to pass the challenge only if they're sufficiently capable of communicating with each other.

Definition 2.2 (Classical zero-error channel coding game)**.** Let $[k] = \{0, 1, \ldots, k-1\}$.

- Charlie generates $i \in [k]$ uniformly at random and sends it to Alice.
- Alice applies an encoding function $\mathrm{Enc} : [k] \to A$.
- Alice sends $\mathrm{Enc}(i)$ over the channel, so that Bob receives a random element $T(\mathrm{Enc}(i))$.
- Bob applies a decoding $\mathrm{Dec} : B \to I$ and sends $i' = \mathrm{Dec}(T(\mathrm{Enc}(I)))$ to Charlie.
- Charlie checks whether $i' = i$.

We say Alice and Bob *win* if they pass the final check. We're primarily interested in the question: when is it possible for Alice and Bob to win with probability 1? (This probability is taken both over Charlie's choice of i and the probability inherent in the map T.) Call the greatest k for which Alice and Bob can win this game with certainty the *zero-error one-shot capacity* of the channel T. (Sometimes the logarithm of this quantity is called the capacity. We won't do any numerics in this article, so the distinction is unimportant for now.)

Example 2.3. Let $A = B = [n]$ and T be the zero-error channel $T(a, b) = \delta_{ab}$. In other words, T is deterministic and always gives Bob the exact message that Alice intended to send. Then Alice and Bob can win the game iff $k \leq n$.

Definition 2.4 (Classical confusability graph)**.** The confusability graph of the channel $T : A \times B \to [0, 1]$ has vertex set A and edge set

$$H_T = \{(a, a') \mid \exists b T(a, b)T(a', b) > 0\}.$$

Intuitively, we say that two of Alice's messages a, a' are *confusable* if there's some message b that Bob could receive such that he's not sure whether Alice intended to send a or a'. The confusability graph captures exactly this notion. If two messages are not confusable, then they are *distinguishable*. To be precise, if Bob knows a priori that Alice will send one of two distinguishable messages, then a posteriori he will always be able to figure out which one she sent. This notion of distinguishability provides a proof for the following:

Proposition 2.5. *Alice and Bob have a winning strategy in the channel coding game with channel T iff H_T has an independent set of size k.*

For the example channel that we defined at the outset, it turns out that the confusability graph is the 5-cycle. Then by Proposition 2.5, the zero-error one-shot channel capacity of this channel is 2. Notice that this is the same capacity as the channel which sends one bit without error. However, it can be shown that if Alice and Bob are allowed multiple uses of the channel, then they can in some sense achieve a capacity of $\frac{5}{2}$. The full story behind this is best saved for another day, but it points to a crucial observation by comparing to the $k = 2$ case of example 2.3.

Remark 2.6. The zero-error one-shot capacity of a channel does not fully characterize the usefulness of the channel for zero-error information-processing tasks.

In order to resolve this weakness, we'll need to generalize our game.

2.2 Classical Source-Channel Coding

First, let's put Proposition 2.5 in a slightly more arrow-theoretic light.

Definition 2.7 (Graph homomorphism, clique)**.** Suppose G and H are graphs and $f : V_G \to V_H$ is a function between them. We say that f is a *graph homomorphism* if it takes edges to edges, i.e.

$$(x, y) \in G \Rightarrow (f(x), f(y)) \in H. \tag{1}$$

We write $G \to H$ (read: "G homs to H") if there exists some homomorphism between G and H. A *clique* in a graph G is a collection of vertices $C \subseteq V_G$ that are pairwise edge-connected, i.e. for all $x \neq y \in C$, $(x, y) \in G$.

Notice that an independent set in a graph H is the same as a clique in the *complement graph* $\overline{H} = \{(a, a') \,|\, (a, a') \notin H\}$, which is the same as a homomorphism $K_n \to \overline{H}$. (If this is your first time seeing graph homomorphisms, take some time to convince yourself of this.)

How can we generalize the game so that graphs other than K_n appear as the source of this homomorphism? In our original channel coding game, Charlie gave her test only to Alice. Let's suppose for a moment that Charlie is Nature, and the data that he gives to Alice are the results of some experiment in Alice's lab. In Bob's lab he has very powerful detectors pointed at Alice's lab. Given that Bob already has some information about Alice's experiment, how much more communication do they need in order for Bob to know the full results?

Proposition 2.8 (Operational interpretation, I)**.** *Alice and Bob win the source-channel coding game iff $G \to \overline{H}$, where G is the characteristic graph of the source and H is the confusability graph of the channel.*

Proof idea. Two vertices are connected by an edge in G iff it's possible that Bob is confused between them after receiving his side information from Charlie. Two vertices are connected in $\overline{H}$ iff they are distinguishable. If there is a homomorphism $G \to \overline{H}$, then this homomorphism resolves all confusion. □

Remark 2.9 (Operational interpretation II). Let H, H' be the confusability graphs of channels T and T'. Suppose that $\overline{H} \to \overline{H}'$. Then any zero-error source channel coding game winnable with T is also winnable with T'.

To make this more concrete, let G be the characteristic graph of some source. The zero-error source-channel coding game on (G, T) is winnable iff $G \to \overline{H}$. But in this case, $G \to \overline{H}'$ by composition, so the zero-error source-channel coding game is also winnable on (G, T'). In other words, the channel T' can succeed at any zero-error communication task that T can succeed at. Furthermore, the homomorphism gives an effective T'-simulation of any zero-error communication protocol using channel T.

This allows us to think of confusability graphs as a kind of "zero-error information processing resource" and graph homomorphisms as a kind of "resource inequality" in precisely the sense studied by Tobias Fritz in [6]. One thing I find interesting about this viewpoint is that the partial order of this resource is pretty complicated: by Welzl's theorem on graph homomorphisms, it is dense! Alas, this is a story for another day.

2.3 Quantum Source-Channel Coding

First, we'll fix our notations for quantum mechanics. If you're unfamiliar and want to learn more, I recommend John Preskill's lecture notes, (`http://www.`

`theory.caltech.edu/people/preskill/ph219/`) Nielsen and Chuang's textbook [10], or the edX course created by Thomas Vidick and Stephanie Wehner. (`https://courses.edx.org/courses/course-v1:CaltechDelftX+QuCryptox+3T2016/info`) Fix Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ for Alice and Bob, respectively. The joint Hilbert space describing Alice and Bob's systems together will be denoted by the tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$.

Definition 2.10 (Quantum state)**.** A *quantum state* $\rho \in \mathcal{L}(\mathcal{H})$ is a positive bounded linear operator with $\operatorname{Tr} \rho = 1$.

For our purposes, quantum states can be thought of as a generalization of probabilitity distributions over finite sets.

Definition 2.11 (Quantum channel)**.** A *quantum channel* T between systems $\mathcal{H}_A$ and $\mathcal{H}_B$ is a completely positive trace-preserving (cptp) map $T : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$.

In the classical case, we look at the stochastic transition matrices because they are the most general linear maps that take probability distributions to probability distributions. Similarly, the cptp maps are the most general linear maps that take quantum states to quantum states.

Theorem 2.12 (Kraus representation)**.** *A linear map $T : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$ is cptp iff there exist* Kraus operators $K_i : \mathcal{H}_A \to \mathcal{H}_B$ *such that $\sum_i K_i^\dagger K_i = I_A$ (where I_A is the identity map on $\mathcal{H}_A$) and the action of T satisfies*

$$T(X) = \sum_i K_i X K_i^\dagger. \tag{2}$$

In general, $|A|\,|B|$ Kraus operators suffice to represent T.

Definition 2.13 (Discrete quantum source-channel coding problem)**.** Charlie has some fixed collection of states $\{\rho^i\}_{i \le k}$ with $\rho_i \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$. Charlie picks i uniformly at random and gives $\rho_A^i = \operatorname{Tr}_B \rho^i$ to Alice and $\rho_B^i = \operatorname{Tr}_A \rho^i$ to Bob. Alice applies some encoding map to ρ_A^i and then sends a message over the channel to Bob. Bob uses $T(\operatorname{Enc}(\rho_A^i))$ and ρ_B^i together to make a guess i'. Alice and Bob win if $i' = i$.

One could consider a slightly more general game where Charlie's choice is a vector from a Hilbert space rather than an element from a finite set. However, we'll see that the discrete game already captures all the phenomena we care about.

In order to characterize the winnable source-channel coding games, we're going to need a fully quantum version of the source characteristic graph, the channel confusability graph, and the graph homomorphism. Noncommutative graphs were first developed by Duan, Severini, and Winter in [5]. The graph homomorphism was introduced by Stahlke [11].

3 Quantum Graphs

Definition 3.1 (Quantum graph)**.** A *quantum graph on* $\mathcal{H}$ is an *operator system* $S \subseteq \mathcal{L}(\mathcal{H})$, i.e. S is a vector space, contains the identity, and is closed under adjoints.

We think of $\mathcal{H}$ as the vertex set and the operators in S as the edge set. We now see that quantum graphs give the right notions to characterize the zero-error quantum source-channel coding game.

Definition 3.2 (From Theorem 14 and Equation 8 of [11])**.** Let T be a channel with Kraus operators $\{K_i\}$. The *confusability graph* of T is $S_T = \text{Span}\left\{K_i^\dagger K_i\right\}$. In the discrete quantum source-channel setup, let $|\psi_i\rangle$ be a purification of $\rho^i \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ for each i, where $\mathcal{H}_C$ is a reference system held by Charlie. Define an isometry $J : \mathbb{C}^k \to \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ by $J = \sum_i |\psi_i\rangle\langle i|$. Then the *characteristic graph* of the source is $S = \text{Tr}_{BC}\left\{\mathcal{L}(\mathcal{H}_E) J K_n J^\dagger\right\}$, where $K_n = \text{Span}\{|i\rangle\langle j| \,|\, i, j \leq n\}$. is the graphical operator system corresponding to the complete graph on n vertices.

I like to think of S_T as the space of "possible error operators" induced by the action of T. Similarly, the characteristic graph of the source is the space of error operators that Alice needs to correct in her communication to Bob in order to win the source-channel coding game.
It turns out that both characteristic graphs and confusability graphs are fully general constructions, even for a seemingly limited class of sources.

Theorem 3.3 (Lemma 2 of [4])**.** *Every quantum graph arises as the confusability graph of some channel.*

Theorem 3.4 (Theorem 17 of [11])**.** *Every quantum graph arises as the characteristic graph of some discrete source with only two inputs.*

In principle, everything that can be studied about quantum graphs could be studied through the lens of zero-error information theory. However, studying them in their own right has already proved somewhat interesting. Nik Weaver has proved a so-called "quantum Ramsey theorem", [12] and Steven Lu has studied a notion of chromatic number that is related to one form of our main conjecture. [9]

Definition 3.5 (Quantum graph homomorphism)**.** Let $S_T \subseteq \mathcal{L}(\mathcal{H}_A)$ and $S_{T'} \subseteq \mathcal{L}(\mathcal{H}_B)$ be quantum graphs. We say that a cptp map $\Phi : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$ is a *quantum graph homomorphism* if $K^\dagger S_{T'} K \subseteq S_T$, where K is the span of the Kraus operators of Φ.

It's not entirely intuitive to interpret this definition. If we again think of $S_{T'}$ as the space of "possible error operators" induced by the action of T', then $K^\dagger S_{T'} K$ is kind of like the space of error operators induced by $\Phi \circ T'$. If this is a subspace of the space of errors induced by T, then T' can simulate T in precisely the sense of remark 2.9. If anyone who knows things about quantum error correction wants to correct me on this point, I'd be grateful.

Nik Weaver thinks of the operator system $K^\dagger SK$ as a "pullback of S along Φ". [13] His point-of-view is bimodule-theoretic and generalizes to infinite dimensions; we don't need that full formalism here. Now we see that quantum graph homomorphisms give us a way to talk about victory in the source-channel coding game.

Theorem 3.6 (Theorem 14 of [11])**.** *Alice and Bob win the quantum source-channel coding game iff $S \to S_T$, where S is the characteristic graph of the source and S_T is the distinguishability graph of the channel.*

As a quick sanity check, we note that the notion of quantum graph extends the notion of classical graph:

Definition 3.7 (Graphical operator system)**.** If G is a classical graph without self-loops, then $S_G = \mathrm{Span}\{|i\rangle\langle j| \mid (i,j) \in G\}^\perp$ is the quantum graph associated with G. If S a quantum graph is equal to S_G, then we'll call S a *graphical operator system.*

Proposition 3.8 (Theorem 8 of [11])**.** *If G and H are graphs, then $S_G \to S_H$ as quantum graphs iff $G \to H$ as classical graphs.*

There are lots and lots of very basic structural questions to ask about the category of quantum graphs with quantum homomorphisms. For now, we'll stick to just computational questions.

Classical graphs are sufficiently expressive to capture all of NP. (I think there is a theorem that makes this precise, but I can't recall it.) Indeed, the literature is rife with computationally interesting problems about classical graph theory.

I'm very interested in the development of a general *quantum graph theory*, a program first suggested in [5]. For the remainder of the article, we'll focus on the following comptuational problem: Given quantum graphs S and S', does $S \to S'$?

4 Complexity of Source-Channel Coding

4.1 Quantum Complexity

In order to talk about the computational complexity of the quantum source-channel coding problem, we'll have to introduce the class QMA, a quantum generalization of NP. I'll assume familiarity with the latter but not the former.

Definition 4.1 (Decision problem)**.** A *language* L is a set of finite strings. The decision problem associated with L is the computational task "given a string x, decide whether $x \in L$."

Definition 4.2 (NP complexity class)**.** A language L (or its associated decision problem) is in NP iff there exists a polynomial time algorithm V (the *verifier*) such that

$$x \in L \Leftrightarrow \exists c V(x, c) \text{ accepts.}$$

c is called the *certificate* for x. We typically think of c as a short proof that $x \in L$.

Promise problems are a relaxation of decision problems that are useful for problems involving real number parameters. Instead of sharply delineating between "in L" and "not in L", we allow for some gray area in between. We are content if our algorithms fail to produce useful answers in this gray area; we can think of the gray area as consisting of instances which require too much numerical accuracy to efficiently distinguish.

Definition 4.3 (Promise problem)**.** A *promise problem* consists of two languages $L_{\text{yes}}, L_{\text{no}}$. The computational task associated to a promise problem is:
Given the promise that $x \in L_{\text{yes}}$ or $x \in L_{\text{no}}$, decide which is the case.

A decision problem can be thought of as a promise problem where $L_{\text{yes}} \cup L_{\text{no}}$ is the set of all strings. It's tempting to define the class QNP that comes from replacing V by a quantum algorithm and c by a quantum certificate. However, quantum algorithms are inherently probabilistic, so we like to allow for positive probability of error.

Definition 4.4 (QMA complexity class)**.** A language L is in QMA iff there exists a polynomial time quantum algorithm V (the *verifier*) such that

$$x \in L \Leftrightarrow \exists c V(x, c) \text{ accepts}$$

Definition 4.5 ((NP)(QMA)-completeness)**.** A language L is NP-complete if for every language L' in NP, there is a polynomial-time reduction R such that $R(x) \in L$ iff $x \in L'$. QMA completeness is defined analogously.

4.2 Classical source-channel coding

Consider first the problem of fixed-source channel coding. We fix some behavior for Charlie in the source-channel coding game, and then ask: given a description of a classical channel T, can Alice and Bob win in the source-channel coding game with Charlie and T? By theorem 3.6, this is equivalent to the following problem.

Definition 4.6 (Fixed-source channel coding problem)**.** Fix a graph X. Given a graph G, does $X \to G$?

Proposition 4.7. *For any X, there is a polynomial time algorithm to decide the fixed-source channel coding problem.*

Proof. The following algorithm works: iterate through all functions $X \to G$. Check if each one is a homomorphism. This runs in time polynomial in $|G|$, the size of the vertex set of G, since there are only $|G|^{|X|}$ functions, where $|X|$ is constant. Checking that a particular function is a homomorphism requires only checking one constraint for each edge of G. □

Definition 4.8 (Fixed-channel source coding problem)**.** Fix a graph X. Given a graph G, does $G \to X$?

Fact 4.9. *Fixed-channel source coding is* NP*-complete, even for X the triangle graph.*

This is the very well known 3-COLORING problem.

4.3 The quantum clique problem

Now we focus on the idea of taking problems on classical graphs and lifting them to quantum graphs. Salman Beigi and Peter Shor had this idea in 2007, 4 years before the introduction of noncommutative graphs! In [1], they defined a problem they called QUANTUM-k-CLIQUE and proved it was QMA-complete. However, they don't take much notice of the fact that their proof works for the case where k is fixed to 2; they had thought of k as an input to the problem. At that time, they were thinking of this as a generalization of an NP-hard problem like MAX-$CLIQUE$. However, we'll argue that it's really a generalization of the very easy problem "Given a graph G, does G have any edges?"

Definition 4.10 (Definition 2.9 of [1],QUANTUM-k-CLIQUE)**.** Given a channel T, do there exist $\rho_1, \ldots, \rho_k$ such that $\sum_{i \neq j} \mathrm{Tr}(T\rho_i)(T\rho_j) \leq a$, or is it instead the case that for all $\rho_1, \ldots, \rho_k$ we have $\sum_{i \neq j} \mathrm{Tr}(T\rho_i)(T\rho_j) \geq b$?

Proposition 4.11. *T is a yes instance of* QUANTUM-k-CLIQUE *with $a = 0$ iff there is a homomorphism $K_k \to S_T$*

(We note that it is justifiable to think about setting a to 0 in QUANTUM-k-CLIQUE: Beigi and Shor prove that this problem is complete for the class QMA_1 of QMA problems with one-sided error. We'll talk only about the QMA case here, but it's easy to see that essentially the same proof works for the QMA_1 case.)

Proof sketch. To simplify notation, let $k = 2$. We go through the discrete source-channel coding game. Let the source send two perfectly distinguishable states $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$ to Alice, while sending nothing to Bob. This source has characteristic graph K_2, which is exactly the classical graph with two vertices and one edge.

Alice and Bob win the source-channel coding game precisely iff Alice has some function $\Phi : |0\rangle\langle 0| \mapsto \rho_1, |1\rangle\langle 1| \mapsto \rho_2$ such that Bob can distinguish between $T\rho_1$ and $T\rho_2$. The map Φ is a homomorphism $K_2 \to S$ by theorem 3.6. □

Theorem 4.12 (Theorem 3.1 of [1])**.** QUANTUM-2-CLIQUE *is* QMA*-complete.*

The proof I present here is exactly the proof given by Beigi and Shor, up to notational differences. I reproduce it in full detail, rather than just referring you to the paper. I do this because the proof is marvellously short! The simplicity of the proof is evidence that quantum graph problems are very natural representatives of the QMA complexity class.

Proof of Theorem 4.12. Let $H = \sum_i^d H_i$ with $\|H_i\| \leq 1$ be an instance of the (a, b)-LOCAL HAMILTONIAN problem, where H acts on a Hilbert space $\mathcal{H}$. We'll give a reduction to the $(\frac{a^2}{d^2}, \frac{b^2}{d^2})$-QUANTUM-2-CLIQUE problem.

Let $M = I - \frac{1}{d}H$, so that $0 \leq M \leq I$ and $0 \leq \frac{1}{d}H \leq M$. In particular, $\{H \otimes I, M \otimes |0\rangle\langle 0|, M \otimes |1\rangle\langle 1|\}$ is a POVM on the Hilbert space $\mathcal{H} \otimes \mathbb{C}^2$. Now consider the channel $T : \mathcal{L}(\mathcal{H} \otimes \mathbb{C}^2) \to \mathcal{L}(\mathbb{C}^3)$ which makes a measurement with this POVM and reports the classical result:

$$T\rho = \frac{1}{d}(\mathrm{Tr}(H \otimes I)\rho)\,|0\rangle\langle 0| + (\mathrm{Tr}(M \otimes |0\rangle\langle 0|)\rho)\,|1\rangle\langle 1| + (\mathrm{Tr}(M \otimes |1\rangle\langle 1|)\rho)\,|2\rangle\langle 2| \tag{3}$$

First, we assume that all eigenvalues of H are at least b and show that no pair of states form a "$\frac{b^2}{d^2}$-2-clique" in the distinguishability graph of T. More precisely, we assume that $\mathrm{Tr}\, H\sigma \geq b$ for all density matrices $\sigma \in \mathcal{L}(\mathcal{H})$. Then for any $\rho \in \mathcal{L}(\mathcal{H} \otimes \mathbb{C}^2)$, we have $\mathrm{Tr}(H \otimes I)\rho \geq b$. (One way to see this is by first performing a partial trace on the $\mathbb{C}^2$ subsystem and then taking the full trace.) Then for any $\rho_1, \rho_2 \in \mathcal{L}(\mathcal{H} \otimes \mathbb{C}^2)$,

$$\mathrm{Tr}(T\rho_1)(T\rho_2) \geq \left(\frac{1}{d}\,\mathrm{Tr}(H \otimes I)\rho_1\right)\left(\frac{1}{d}\,\mathrm{Tr}(H \otimes I)\rho_2\right) \geq \frac{b^2}{d^2}. \tag{4}$$

We conclude that if H is a no instance of LOCAL HAMILTONIAN , then T is a no instance of QUANTUM-2-CLIQUE.

Next, we assume that H has a small eigenvalue and find a "$\frac{a^2}{d^2}$-2-clique" in the distinguishability graph of T. More precisely, let σ be such that $\mathrm{Tr}\, H\sigma \leq a$. Then let $\rho_1 = \sigma \otimes |0\rangle\langle 0|, \rho_2 = \sigma \otimes |1\rangle\langle 1|$. It's clear that $\mathrm{Tr}(H \otimes I)\rho_1 = \mathrm{Tr}\, H\sigma = \mathrm{Tr}(H \otimes I)\rho_2$, so that in particular each of these are at most a. Furthermore, we have $\mathrm{Tr}(M \otimes |0\rangle\langle 0|)\rho_2 = \mathrm{Tr}(M \otimes |1\rangle\langle 1|)\rho_1 = 0$. Therefore,

$$(T\rho_1)(T\rho_2) = \left(\frac{1}{d}\,\mathrm{Tr}\, H\sigma\right)^2 |0\rangle\langle 0|. \tag{5}$$

In particular, we have

$$\mathrm{Tr}(T\rho_1)(T\rho_2) \leq \frac{a^2}{d^2}. \tag{6}$$

We conclude that if H is a yes instance of LOCAL HAMILTONIAN , then T is a yes instance of QUANTUM-2-CLIQUE. So the map $H \mapsto T$ is a reduction from LOCAL HAMILTONIAN to QUANTUM-2-CLIQUE. Therfore, the latter is QMA-complete. □

We saw that QUANTUM-2-CLIQUE can be thought of as a fixed-source quantum graph homomorphism problem. We now introduce QUANTUM k-COLORING as a fixed-target quantum graph homomorphism problem.

Definition 4.13 (QUANTUM k-COLORING, fully quantum version)**.**

- Input: States $\rho_1, \rho_2 \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$.
- Promise: Either there exists some cptp map $T : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(C^k)$ such that
$$\mathrm{Tr}[(T \otimes I_B)\rho_1][(T \otimes I_B)\rho_2] \leq a, \tag{7}$$
or else for all such T,
$$\mathrm{Tr}[(T \otimes I_B)\rho_1][(T \otimes I_B)\rho_2] \geq b. \tag{8}$$
- Problem: Decide which is the case.

Think of $(T \otimes I_B)\rho_i$ as the state that Bob holds in the source-channel coding game where Alice uses encoding map T and then communicates with the identity channel $I : \mathcal{L}(\mathbb{C}^k) \to \mathcal{L}(\mathbb{C}^k)$.

Proposition 4.14. *ρ_1, ρ_2 is a yes instance of* QUANTUM-k-CLIQUE *with $a = 0$ iff there is a homomorphism $S \to \mathcal{L}(\mathbb{C}^k)$, where S is the characteristic graph of the discrete source which produces ρ_1 or ρ_2.*

The proof of this is essentially the same as the proof of proposition 4.11. If we give Alice and Bob only a classical communication channel, the problem changes a bit.

Definition 4.15 (QUANTUM k-COLORING, quantum-classical version)**.**

- Input: States $\rho^1, \rho^2 \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$.
- Promise: Either there exists some POVM $\{M_i\}$ on $\mathcal{H}_A$ such that when Alice measures ρ^i_A with $\{M_i\}$ and communicates the outcome to Bob, he can guess i with probability $1 - a$, OR

  for any POVM that Alice chooses, Bob's guessing probability is at most $\frac{1}{2} + b$.
- Problem: Decide which is the case.

The POVM $\{M_i\}$ induces the cptp map $T\rho = \sum_I (\mathrm{Tr}\, M_i \rho) |i\rangle\langle i|$. This is precisely a homomorphism to K_k.

Proposition 4.16. *ρ_1, ρ_2 is a yes instance of* QUANTUM-k-CLIQUE *with $a = 0$ iff there is a homomorphism $S \to K_k$, where S is the characteristic graph of the discrete source which produces ρ_1 or ρ_2.*

Problem 4.1. Classify the computational complexity of QUANTUM k-COLORING.

It's immediate that the latter formulation is NP-hard, since classical graphs and classical graph homomorphisms are a special case of quantum graphs and quantum graph homomorphisms. It's not hard to show that both formulations are in QMA: the witness is the homomorphism. Following our intuition in the classical case, this problem smells like it should be harder than QUANTUM-k-CLIQUE and therefore QMA-complete. Of course, it's also plausible that there's some clever algorithm putting this in BQP. Any thoughts towards a resolution of this problem are appreciated.

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